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An Overview of Recent Developments of Fuzzy Divergence Measures and their Generalizations

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ABSTRACT

Aim: The aim of the paper is to present an overview of recent developments in fuzzy divergence measures.

Methodology: We analysis some of well-known notions and the concepts associated with fuzzy set theory with an axiomatic definition of fuzzy divergence measure.

Results: Fuzzy divergence measures and their generalizations existing in the literature are summarized briefly.

Conclusion: The reviewed divergence measures and their generalizations are useful for further development of measures of divergence and appropriate to deal with different areas.

Key Words: Fuzzy Sets, Fuzzy Measure of Information, Fuzzy Divergence Measures

INTRODUCTION

In 1965, as a generalization of classical (crisp) set theory, Zadeh [1] first proposed the concept of fuzzy set theory. It was found to be suitable in dealing with many fields like pattern recognition, image processing, speech recognition, bio-informatics, fuzzy aircraft control, feature selection, decision making, etc. Over the last six decades, the study on fuzzy set (FS) theory and its application to different area has been extended evenly by the researchers. Afterwards, in 1965, Zadeh [2] introduced the notion of fuzzy entropy as a measure of uncertainty. In 1972, De Luca and Termini [3] presented an axiomatic composition of fuzzy entropy corresponding to the probabilistic entropy measure of Shannon [4]. For determining the difference between the two FSs divergence measure is a significant tool among the most exciting measures in FSs theory. The concept of fuzzy divergence measure is itself one of the generalization of fuzzy information measure. Analogous to Kullback and Leibler [5] measure of divergence, Bhandari and Pal [6] introduced the fuzzy divergence measure. Thereafter literature on fuzzy measures of information, divergence and their generalizations are considerably extended by the different researchers [7-25] in the past decades.

Briefly motivated by the study of the literature on measures of fuzzy information and their generalization in the paper Methodology section introduce some well-known concepts, and the notation related to fuzzy set theory, axiomatic definitions of fuzzy information measure, fuzzy divergence measure etc. In the results section, we provide an overview of recent developments in fuzzy divergence measures and their generalizations. An essential analysis of the existing measures of divergence is presented in the discussion section. The final section concludes the paper.

METHODOLOGY

We now introduce some of well-known notions and the concepts associated with fuzzy set theory with an axiomatic definition of fuzzy divergence measure.

Zadeh [1] defined a fuzzy set (FS) X on a universe of discourse U having the membership function $\mu_X : U \to [0,1]$ as follows:

$$X = \{\langle a, \mu_X(a)\rangle / a \in U\}$$

The membership value $\mu_X(a)$ describes the degree of the belongingness of $a \in U$ in *X*. When $\mu_X(a)$ is valued in {0,

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1}, it is the characteristic function of a crisp (i.e., non-fuzzy) set.

The measure of difference between two fuzzy sets is called the fuzzy divergence measure.

Bhandari and Pal [6] initiated the measure of fuzzy divergence corresponding to divergence measure of Kullback and Leibler [5], as

$$I(A:B)=\sum_{i=1}^{n}\left[\mu_A(a_i)\log\frac{\mu_A(a_i)}{\mu_B(a_i)}+(1-\mu_A(a_i))\log\frac{1-\mu_A(a_i)}{1-\mu_B(a_i)}\right] \quad (1)$$

satisfying the conditions:

$$I(A:B)\geq 0$$

$$I(A:B)=0 \text{ if } A=B$$

$I(A:B)$ is a convex function of $\mu_A(a_i)$.

Bhandari and Pal [6] also defined the fuzzy symmetric divergence measure:

$$J(A:B)=I(A:B)+I(B:A)=\sum_{i=1}^{n}\left[(\mu_A(x_i)-\mu_B(x_i))\log\frac{\mu_A(x_i)(1-\mu_B(x_i))}{\mu_B(x_i)(1-\mu_A(x_i))}\right] \quad (2)$$

RESULTS

A general study of the axiomatic definition of a divergence measure for fuzzy sets was also presented in Bouchon-Meunier et al. [26]. As a significant content in fuzzy mathematics, the research on divergence measures between fuzzy sets has received more attention. In recent years, some definitions of generalized measures of fuzzy divergence have been proposed.

Afterwards, Shang and Jiang [27] provided a modified version of the fuzzy divergence measure of Bhandari and Pal [6] with the novel idea of Lin [28] and defined as

$$D(A:B)=\frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(a_i)\log\frac{\mu_A(a_i)}{\left(\frac{\mu_A(a_i)+\mu_B(a_i)}{2}\right)}+(1-\mu_A(a_i))\log\frac{1-\mu_A(a_i)}{\left(\frac{2-\mu_A(a_i)-\mu_B(a_i)}{2}\right)}\right] \quad (3)$$

Fan and Xie [29] proposed the fuzzy information of discrimination of A against B corresponding to the exponential fuzzy entropy of Pal and Pal [30] and is given by

$$I(A:B)=\sum_{i=1}^{n}\left[1-(1-\mu_A(a_i))e^{(\mu_A(a_i)-\mu_B(a_i))}-\mu_A(a_i)e^{(\mu_B(a_i)-\mu_A(a_i))}\right] \quad (4)$$

Thereafter Couso et al. [31] define that if X is a universe of discourse and $F(X)$ is the set of all fuzzy subsets, a mapping $D:F(X)\times F(X)\to R$ is a divergence measure between fuzzy subsets if and only if for each $A,B,C\in F(X)$, the following axioms hold:

$d_1: D(A:B)=D(B:A)$

$d_2: D(A:A)=0$

$d_3: \max\{D(A\cup C:B\cup C), D(A\cap C:B\cap C)\}\leq D(A:B)$

Non-negativity of $D(A:B)$ is the natural assumption.

Montes et al. [32] studied the special classes of divergence measures and used the link between fuzzy and probabilistic uncertainty. It also studied widely the divergence measure for fuzzy sets as a particular case.

Hooda [33] presented a fuzzy divergence measure corresponding to Havada-Charvat [34] measure of divergence which is given by

$$I_\alpha(X,Y)=\frac{1}{\alpha-1}\sum_{i=1}^{n}[\mu_X^\alpha(b_i)\mu_Y^{1-\alpha}(b_i)+(1-\mu_X(b_i))^\alpha(1-\mu_Y(b_i))^{1-\alpha}-1],\ \alpha\neq 1,\alpha>0. \quad (5)$$

Parkash et al. [35] proposed a fuzzy divergence measure corresponding to Ferreri [36] probabilistic measure of divergence given by

$$\overline{I_{\alpha 1}}(A:B)=\frac{1}{a}\sum_{i=1}^{n}\left[(1+a\mu_A(y_i))\log\frac{1+a\mu_A(y_i)}{1+a\mu_B(y_i)}+\{1+a(1-\mu_A(y_i))\}\log\frac{1+a(1-\mu_A(y_i))}{1+a(1-\mu_B(y_i))}\right]; a>0 \quad (6)$$

and another one given by Parkash et al. [35] is

$$I_{a2}(A:B)=\sum_{i=1}^{n}\left[\mu_A(y_i)\log\frac{\mu_A(y_i)}{\mu_B(y_i)}+(1-\mu_A(y_i))\log\frac{1-\mu_A(y_i)}{1-\mu_B(y_i)}\right]$$
$$-\frac{1}{a}\sum_{i=1}^{n}\left[(1+a\mu_A(y_i))\log\frac{1+a\mu_A(y_i)}{1+a\mu_B(y_i)}+\{1+a(1-\mu_A(y_i))\}\log\frac{1+a(1-\mu_A(y_i))}{1+a(1-\mu_B(y_i))}\right] \quad (7)$$

Corresponding to Renyi [36] and Sharma and Mittal [37] generalized measure of divergence Bajaj and Hooda [38] provided the generalized fuzzy divergence measure which are given by

$$D_\beta(A,B)=\frac{1}{\beta-1}\sum_{i=1}^{n}\log[\mu_A^\beta(z_i)\mu_B^{1-\beta}(z_i)+(1-\mu_A(z_i))^\beta(1-\mu_B(z_i))^{1-\beta}],\ \beta\neq 1,\beta>0. \quad (8)$$

and $D_{\alpha,\beta}(X,Y)$

$$=\frac{1}{2^{1-\beta}-1}\sum_{i=1}^{n}[(\mu_X^\alpha(z_i)\mu_Y^{1-\alpha}(z_i)+(1-\mu_X(z_i))^\alpha(1-\mu_Y(z_i))^{1-\alpha})^{\frac{\beta-1}{\alpha-1}}-1],$$

$$\alpha\neq 1,\alpha>0,\beta\neq 1,\beta>0. \quad (9)$$

DISCUSSION

Singh and Tomar [39, 40,41,42] have defined and studied some of symmetric and non-symmetric fuzzy divergence measures analogous to probabilistic divergence measures and inequalities among them. Moreover, Singh and Tomar [43,44] presented a number of refinement of inequalities among fuzzy divergence measures. Bhatia and Singh [45] proposed three families of fuzzy divergence.

Thereafter, Hooda and Jain [46] presented a generalized fuzzy divergence measure, its ambiguity and information improvement. Further, in the recent years, fuzzy measures of information, divergence and their generalizations are considerably extended by Tomar and Ohlan [17-20] and Ohlan and Ohlan [9-16] and Ohlan [7,8, 21-25] and find their applications emerging in various fields like decision making (e.g., multi-criteria decision making, multi-attribute decision making), pattern recognition, medical diagnosis and suitability in linguistic variables.

CONCLUSION

In this paper we have presented a review of the fuzzy measures of divergence and their generalizations existing in the literature. It was noted that these measures of fuzzy divergence can be used for further development of generalized fuzzy divergence measures, appropriate to deal with different areas.

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