ijcrr
Vol 04 issue 12
Category: Research
Received on:21/04/12
Revised on:05/05/12
Accepted on:16/05/12

# BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS UNDER G-BROWNIAN MOTION WITH DISCONTINUOUS DRIFT COEFFICIENTS

Faiz Faizullah[1,2], Rahman Ullah[3]

[1]College of Physical and Environmental Oceanography, Ocean University of China, Qingdao, PR China
[2]College of Electrical and Mechanical Engineering (EME), National University of Sciences and Technology (NUST), Islamabad, Pakistan.
[3]Department of Mathematics, Abdul Wali Khan University Mardan, Pakistan

E-mail of Corresponding Author: faiz_math@yahoo.com

## ABSTRACT

The main objective of this paper is to introduce the upper and lower solutions method for backward stochastic differential equations under G-Brownian motion (G-BSDEs). The existence of solutions for backward stochastic differential equations under G-Brownian motion having a discontinuous drift coefficient is shown with the method of upper and lower solutions. As an example, a scalar stochastic differential equation under G-Brownian motion having the unit step function as a drift coefficient is considered.

**Key words:** Upper and lower solutions, backward stochastic differential equations**,** G-Brownian motion**,** discontinuous drift coefficient**,** existence**.**

## INTRODUCTION

To measure super hedging and risk in finance under volatility uncertainty, the G-Brownian motion and the related stochastic calculus were introduced by Peng [16, 17]. He introduced the backward stochastic differential equations under G-Brownian motion (G-BSDEs) and developed the existence and uniqueness of solutions for G-BSDEs with Lipschitz continuous coefficients, see [18] chapter IV page 83 or the appendix of this paper. Later, X. Bai and Y. Lin extended the existence and uniqueness theory of the G-BSDEs to the integral Lipschitz coefficients [2]. Also see [20] for the stability theorem of G-BSDEs.

Now here in contrast to the above, we introduce the method of upper and lower solutions and establish the existence theory for G-BSDEs with discontinuous drift coefficients, such as in the following scalar G-BSDE

$$X_t = E[\xi + \int_t^T u(X_v)dv + \int_t^T d\langle B\rangle_v \,|\, \Omega_t],\ \ t \in [0,\ T],$$

where the Heaviside function or the unit step function $u : R \to R$, defined by

$$u(x) = \begin{cases} 0, & \text{if } x < 0\,; \\ 1, & \text{if } x \geq 0, \end{cases}$$

is an important function in science and engineering. This function arises in many discontinuous ordinary differential equations [11].

We study the following backward stochastic differential equation under G-Brownian motion

$$X_t = E[\xi + \int_t^T f(v, X_v)dv + \int_t^T g(v, X_v)d < B >_v | \Omega_t], \quad t \in [0, T], \tag{1.1}$$

where $\xi \in L_G^1(\Omega_T,\ R^n)$ is given, $\{\langle B\rangle_t : t \geq 0\}$ is the quadratic variation process of one dimensional (only for simplicity) G-Brownian motion $\{B_t : t \geq 0\}$ and the coefficients

$f(t,\ x),\ \ g(t,\ x)\in M_G^1(0,\ T;\ R^n).$ It is assumed that $f(t,x)$ is a discontinuous function where $g(t,x)$ is Lipschitz continuous for all $x\in R^n$. A process $X_t$ belongs to the mentioned space satisfying the G-BSDE (1.1) is said to be its solution.

This paper is organized as follows. In section 2 some basic definitions and notions of the G-expectation are given. In section 3 the upper and lower solutions method for G-BSDEs is introduced. In section 4 the comparison theorem is established while the existence of solutions for the G-BSDEs with discontinuous drift coefficient is shown in section 5. Appendix is given in section 6.

## PRELIMINARIES

In this section some basic definitions and mathematical results, which will be used for our later pursuit are given [4, 7, 14, 15, 19].

Let $\Omega$ be a (non-empty) basic space and $\mathbf{H}$ be a linear space of real valued functions defined on $\Omega$ such that any arbitrary constant $c\in\mathbf{H}$ and if $X\in\mathbf{H}$ then $|X|\in\mathbf{H}$. We consider that $\mathbf{H}$ is the space of random variables.

**Definition 2.1.** A functional $E:\mathbf{H}\to R$ is called sublinear expectation, if $\forall X,Y\in\mathbf{H}, c\in R$ and $\lambda\geq 0$ it satisfies the following properties

(1) (Monotonicity): If $X\geq Y$ then $E[X]\geq E[Y]$.

(2) (Constant preserving): $E[c]=c$.

(3) (Sub-additivity): $E[X+Y]\leq E[X]+E[Y] or E[X]-E[Y]\leq E[X-Y]$.

(4) (Positive homogeneity): $E[\lambda X]=\lambda E[X]$.

The triple $(\Omega,\ \mathbf{H},\ E)$ is called a sublinear expectation space. Consider the space of random variables $\mathbf{H}$ such that if $X_1,\ X_2,...,\ X_n\in\mathbf{H}$ then $\varphi(X_1,\ X_2,...,\ X_n)\in\mathbf{H}$ for each $\varphi\in C_{l.Lip}(R^n)$, where $C_{l.Lip}(R^n)$ is the space of linear functions $\varphi$ defined as the following

$$C_{l.Lip}(R^n)=\{\varphi:R^n\to R\,|\,\exists C\in R^+, m\in N$$
$$s.t.|\varphi(x)-\varphi(y)|\leq c(1+|x|^m+|y|^m)|x-y|\},$$

for $x,\ y\in R^n$.

**Definition 2.2.** Two n-dimensional random vectors $X$ and $\hat{X}$ defined respectively on the sublinear expectation spaces $(\Omega,\ \mathbf{H},\ E)$ and $(\hat{\Omega},\ \hat{\mathbf{H}},\ \hat{E})$ are said to be identically distributed if

$$E[\varphi(X)]=\hat{E}[\varphi(\hat{X})],\quad \forall\quad \varphi\in C_{l.Lip}(R^n).$$

$X$ and $\hat{X}$ are identically distributed is expressed by $X\sim\hat{X}$.

**Definition 2.3.** Let $(\Omega,\ \mathbf{H},\ E)$ be a sublinear expectation space and $X\in\mathbf{H}$ with

$\bar{\sigma}^2=E[X^2],\underline{\sigma}^2=-E[-X^2].$

Then $X$ is said to be G-normally distributed or $N(0;[\bar{\sigma}^2,\ \underline{\sigma}^2])$ -distributed, if $\forall a,b\geq 0$ we have

$aX+bY\sim\sqrt{a^2+b^2}X,$

for each $Y\in\mathbf{H}$ which is independent to $X$ and $Y\sim X$.

**G-expectation and G-Brownian Motion.** Let $\Omega=C_0(R^+)$ , that is, the space of all $R$ -valued continuous paths $(w_t)_{t\in R^+}$ with $w_0=0$ equipped with the distance

$$\rho(w^1,\ w^2)=\sum_{k=1}^{\infty}\frac{1}{2^k}(\max_{t\in[0,\ k]}|w_t^1-w_t^2|\wedge 1),$$

and consider the canonical process $B_t(w)=w_t$ for $t\in[0,\infty),\quad w\in\Omega$ then for each fixed $T\in[0,\infty)$ we have

$$L_{ip}(\Omega_T)=\{\varphi(B_{t_1},\ B_{t_2},...,\ B_{t_n}):t_1,...,\ t_n\in[0,\ T],$$
$$\varphi\in C_{l.Lip}(R^n),\ n\in N\},$$

where $L_{ip}(\Omega_t)\subseteq L_{ip}(\Omega_T)$ for $t\leq T$ and $L_{ip}(\Omega)=\cup_{m=1}^{\infty}L_{ip}(\Omega_m)$.

Consider a sequence $\xi_i{}_{i=1}^{\infty}$ of n-dimensional random vectors on a sublinear expectation space $(\hat{\Omega},\ \hat{\mathbf{H}},\ \hat{E})$ such that $\xi_i$ is G-normally distributed and $\xi_{i+1}$ is independent of $(\xi_1,\ \xi_2,...,\ \xi_i)$ for each $i=1,\ 2,...,\ n-1.$ Then a sublinear expectation $E$ defined on $L_{ip}(\Omega)$ is introduced as follows.

For $0=t_0<t_1<...<t_n<\infty,\quad \varphi\in C_{l.Lip}(R^n)$ and each

$X=\varphi(B_{t_1}-B_{t_0},\ B_{t_2}-B_{t_1},...,\ B_{t_n}-B_{t_{n-1}})\in L_{ip}(\Omega),$

$$E[\varphi(B_{t_1}-B_{t_0},\ B_{t_2}-B_{t_1},...,\ B_{t_n}-B_{t_{n-1}})]$$
$$=\hat{E}[\varphi(\sqrt{t_1-t_0}\xi_1,...,\ \sqrt{t_n-t_{n-1}}\xi_n)]$$

The conditional sublinear expectation of $X\in L_{ip}(\Omega_t)$ is defined by

$$E[X\mid\Omega_t]=E[\varphi(B_{t_1},B_{t_2}-B_{t_1},\ B_{t_3}-B_{t_2},...,\ B_{t_m}-B_{t_{m-1}})\mid\Omega_t]$$

$$=\psi(B_{t_1},B_{t_2}-B_{t_1},\ B_{t_3}-B_{t_2},...,\ B_{t_j}-B_{t_{j-1}}),$$

where

$$\psi(x_1,...,\ x_j)=\hat{E}[\varphi(x_1,...,\ x_j,\ \sqrt{t_{j+1}-t_j}\xi_{j+1},...,\ \sqrt{t_n-t_{n-1}}\xi_n)].$$

**Definition 2.4.** The sublinear expectation $E$ defined above is called a G-expectation and the corresponding canonical process $(B_t)_{t\geq 0}$ is called a G-Brownian motion.

The completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p=(E[|X|^p])^{1/p}$ for $p\geq 1$ is denoted by $L_G^p(\Omega)$ and $L_G^p(\Omega_t)\subseteq L_G^p(\Omega_T)\subseteq L_G^p(\Omega)$ for $0\leq t\leq T<\infty$. The filtration generated by the canonical process $(B_t)_{t\geq 0}$ is denoted by $\mathbf{F}_t=\sigma\{B_s,\ 0\leq s\leq t\}$, $\mathbf{F}=\{\mathbf{F}_t\}_{t\geq 0}$.

**Itô Integral of G-Brownian motion.** For any $T\in R^+$, a finite ordered subset $\Pi_T=\{t_0,\ t_1,...,\ t_N\}$ such that $0=t_0<t_1<...<t_N=T$ is a partition of $[0,\ T]$ and

$$\mu(\Pi_T)=\max\{|t_{i+1}-t_i|:\ i=0,\ 1,...,\ N-1\}.$$

A sequence of partitions of $[0,\ T]$ is denoted by $\Pi_T^N=\{t_0^N,\ t_1^N,...,\ t_N^N\}$ such that $\lim_{N\to\infty}\mu(\Pi_T^N)=0.$

Consider the following simple process:

Let $p\geq 1$ be fixed. For a given partition $\Pi_T=\{t_0,\ t_1,...,\ t_N\}$ of $[0,\ T]$,

$$\eta_t(w)=\sum_{i=0}^{N-1}\xi_i(w)I_{[t_i,\ t_{i+1}]}(t), \qquad (2.1)$$

where $\xi_i\in L_G^p(\Omega_t)$, $i=0,\ 1,...,\ N-1$. The collection containing the above type of processes, that is, containing $\eta_t(w)$ is denoted by $M_G^{p,0}(0,\ T)$. The completion of $M_G^{p,0}(0,\ T)$ under the norm $\|\eta\|=\{\int_0^T E[|\eta_v|^p]dv\}^{1/p}$ is denoted by $M_G^p(0,\ T)$ and for $1\leq p\leq q$, $M_G^p(0,\ T)\supset M_G^q(0,\ T).$

**Definition 2.5.** For each $\eta_t\in M_G^{2,0}(0,\ T)$, the Itô's integral of G-Brownian motion is defined as

$$I(\eta)=\int_0^T\eta_v dB_v=\sum_{i=0}^{N-1}\xi_i(B_{t_{i+1}}-B_{t_i}),$$

where $\eta_t$ is given by (2.1).

**Definition 2.6.** An increasing continuous process $\{\langle B\rangle_t:\ t\geq 0\}$ with $\langle B\rangle_0=0$, defined by

$$\langle B\rangle_t=\sum_{i=0}^{N-1}(B_{t_{i+1}^N}-B_{t_i^N})^2=B_t^2-2\int_0^t B_v dB_v,$$

is called the quadratic variation process of G-Brownian motion.

For the details of the following two definitions see [4, 18].

**Definition 2.7.** Let $\mathbf{B}(\Omega)$ be the Borel $\sigma$-algebra of $\Omega$ and $\mathbf{P}$ be a (weakly compact) collection of probability measures $P$ defined on $(\Omega,\ \mathbf{B}(\Omega))$ then the capacity $\hat{c}(.)$ associated to $\mathbf{P}$ is defined by

$$\hat{c}(A)=\sup_{P\in\mathbf{P}}P(A), \qquad A\in\mathbf{B}(\Omega).$$

**Definition 2.8.** A set $A$ is said to be polar if its capacity is zero, that is, $\hat{c}(A)=0$ and a property holds "quasi-surely" (q.s in short) if it holds outside a polar set.

Through out the paper for $X=(x_1,\ x_2,...,\ x_n)$, $Y=(y_1,\ y_2,...,\ y_n)$, $X\leq Y$ means $x_i\leq y_i$, $i=1,\ 2,...,\ n.$

## TME METHOD OF UPPER AND LOWER SOLUTIONS FOR G-BSDEs

Recall [1, 8, 9, 13] for the concept of upper and lower solutions in the sense of classical stochastic differential equations (SDEs).

**Definition 3.1.** A process $L_t\in M_G^1(0,\ T;\ R^n)$ is said to be a lower solution of the G-BSDE (1.1) on the interval $[0,\ T]$ if for any fixed $S$ the inequality (interpreted component wise)

$$L_t\leq E[L_S+\int_t^S f(v,L_v)dv+\int_t^S g(v,L_v)d\langle B\rangle_v\mid\Omega_t\mid,0\leq t\leq S\leq T, \qquad (3.1)$$

holds q.s.

**Definition 3.2.** A process $U_t\in M_G^1(0,\ T;\ R^n)$ is said to be an upper solution of the G-BSDE (1.1) on the interval $[0,\ T]$ if for any fixed $S$ the inequality (interpreted component wise)

$$U_t \geq E[U_S + \int_t^S f(v,U_v)du + \int_t^S g(v,U_v)d\langle B\rangle_v \mid \Omega_t], 0 \leq t \leq S \leq T, \qquad (3.2)$$

holds q.s.

**Example 3.3.** Consider the following scalar G-BSDE

$$X_t = E[\xi + \int_t^T u(X_v)dv + \int_t^T d\langle B\rangle_v \mid \Omega_t], \ t \in [0, \ T], \qquad (3.3)$$

where $\xi$ is a given terminal condition. Then $U_t = E[\xi + \int_t^T dv + \int_t^T d\langle B\rangle_v \mid \Omega_t]$ and $L_t = E[\xi + \int_t^T d\langle B\rangle_v \mid \Omega_t]$ for $t \in [0, \ T]$ are the respective upper and lower solutions for the G-BSDE (3.3) shown below.

Since

$$U_t = E[\xi + \int_t^T dv + \int_t^T d\langle B\rangle_v \mid \Omega_t] = E[U_S + \int_t^S dv + \int_t^S d\langle B\rangle_v \mid \Omega_t]$$

where $U_S = E[\xi + \int_S^T dv + \int_S^T d\langle B\rangle_v \mid \Omega_t]$ for any fixed $S$ such that $0 \leq t \leq S \leq T$. Also

$$U_S + \int_t^S dv + \int_t^S d\langle B\rangle_v \geq U_S + \int_t^S u(U_v)dv + \int_t^S \{U_v\} d\langle B\rangle_v,$$

which implies that

$$E[U_S + \int_t^S dv + \int_t^S d\langle B\rangle_v \mid \Omega_t]$$
$$\geq E[U_S + \int_t^S u(U_v)dv + \int_t^S \{U_v\} d\langle B\rangle_v \mid \Omega_t], \qquad 0 \leq t \leq S \leq T.$$

Hence $U_t = E[\xi + \int_t^T dv + \int_t^T d\langle B\rangle_v \mid \Omega_t]$ for $t \in [0, \ T]$ is the upper solution of the scalar G-BSDE (3.3). On similar arguments one can show that $L_t = E[\xi + \int_t^T d\langle B\rangle_v \mid \Omega_t]$ is the lower solution of the G-BSDE (3.3). The existence of solutions for the G-BSDE (3.3) will be discussed latter in section 5.

Now we assume that $L_t$ and $U_t$ are the respective lower and upper solutions of the G-BSDE

$$X_t = E[\xi + \int_t^T f(v,w)dv + \int_t^T g(v,X_v)d\langle B\rangle_v \mid \Omega_t], \qquad t \in [0,T]. \qquad (3.5)$$

We define two functions $p, \ r : [0, \ T] \times R^n \times \Omega \to R^n$ by

$$p(t, \ x, \ w) = \max\{L_t(w), \ \min\{U_t(w), \ x\}\},$$
$$r(t, \ x, \ w) = p(t, \ x, \ w) - x, \qquad (3.6)$$

and consider the backward stochastic differential equation

$$X_t = E[\xi + \int_t^T \tilde{f}(v,X_v)dv + \int_t^T \tilde{g}(v,X_v)d\langle B\rangle_v \mid \Omega_t], \qquad t \in [0,T], \qquad (3.7)$$

where

$$\tilde{f}(t, \ x, \ w) = f(t, \ w) + r(t, \ x, \ w),$$
$$\tilde{g}(t, \ x, \ w) = g(t, p(x)) \qquad (3.8)$$

are Lipschitz continuous in $x$. It is known that the G-BSDE (3.7) has a unique solution $X_t \in M_G^1(0, \ T; \ R^n)$ see chapter III page 84 of [18] or the appendix. Also see [2].

## COMPARISON RESULT FOR G-BSDEs

The following lemma is very important and will be used in the next comparison theorem.

**Lemma 4.1.** Assume that the respective lower and upper solutions $L_t$ and $U_t$ of the G-BSDE (3.5) satisfy the condition $L_t \leq U_t$ for $t \in [0, \ T]$. Then $L_t$ and $U_t$ are lower and upper solutions of the G-BSDE (3.7) respectively.

**Proof.** Assume that $L_t$ is a lower solution of the G-BSDE (3.5). Since $L_t \leq U_t$ so $p(t, \ L_t) = L_t$ and $r(t, \ L_t) = 0.$ Thus

$$E[L_S + \int_t^S \tilde{f}(v, \ L_v)dv + \int_t^S \tilde{g}(v, \ L_v)d\langle B\rangle_v \mid \Omega_t]$$
$$= E[L_S + \int_t^S [f(v,w) + r(v,L_v)]dv + \int_t^S [g(v, p(v,L_v))d\langle B\rangle_v \mid \Omega_t]$$
$$= E[L_S + \int_t^S f(v,w)dv + \int_t^S [g(v, p(v,L_v))d\langle B\rangle_v \mid \Omega_t] \geq L_t.$$

Hence $L_t$ is a lower solution of G-BSDE (3.7). Also $L_t \leq U_t$ yields $p(t, \ U_t) = U_t$ and $r(t, \ U_t) = 0$. Hence

$$E[U_S + \int_t^S \tilde{f}(v, \ U_v)dv + \int_t^S \tilde{g}(v, \ U_v)d\langle B\rangle_v \mid \Omega_t]$$
$$= E[U_S + \int_t^S [f(v, \ w) + r(v, \ U_v)]dv + \int_t^S g(v, p(v, \ U_v))d\langle B\rangle_v \mid \Omega_t]$$
$$= E[U_S + \int_t^S f(v, \ w)dv + \int_t^S g(v, p(v, \ U_v))d\langle B\rangle_v \mid \Omega_t] \leq U_t.$$

Thus $U_t$ is an upper solution of (3.7).

**Theorem 4.3.** Assume that

(1). The mapping $f$ is measurable with $\int_t^T E[\mid f(v,.)\mid]dv < \infty$ and $g$ is Lipschitz continuous.

(2). The respective lower and upper solutions $L_t$ and $U_t$ of the G-BSDE (3.5) with

$E[|L_t|]<\infty$, $E[|U_t|]<\infty$ satisfy $L_t \le U_t$ for $t\in[0,\ T]$.

(3). Also $X_T=\xi\in L_G^1(\Omega_T,\ R^n)$ is a given terminal value with $E[|X_T|]<\infty$ such that $L_T\le X_T\le U_T$.

Then there exists a unique solution $X_t\in M_G^1(0,\ T;\ R^n)$ of the G-BSDE (3.5) such that $L_t\le X_t\le U_t$ for $t\in[0,\ T]$ q.s.

**Proof.** We consider the functions $p,\ r:[0,\ T]\times R^n\times\Omega\to R^n$ defined by (3.6) and the backward stochastic differential equation (3.7).

Now it is known that the G-BSDE (3.7) has a unique solution and by lemma 4.1 if $L_t$ and $U_t$ are the lower and upper solutions of the G-BSDE (3.5) respectively then they are the respective lower and upper solutions for the G-BSDE (3.7). It is also clear that any solution $X_t$ of the modified G-BSDE (3.7) such that

$$L_t\le X_t\le U_t,\quad t\in[0,\ T], \qquad (4.1)$$

q.s. is also a solution of the G-BSDE (3.5). Thus we only need to show that any solution $X_t$ of the modified problem (3.7) does satisfy the inequality (4.1).

Suppose that there exists an arbitrary interval $(t_1,\ t_2)\subset[0,\ T]$ such that $X_{t_2}=L_{t_2}=\zeta$ and $X_t<L_t$ for $t\in(t_1,\ t_2)$.

Then

$$L_t-X_t=E[\zeta+\int_t^{t_2}\tilde f(v,\ L_v)dv+\int_t^{t_2}\tilde g(v,\ L_v)d\langle B\rangle_v\,|\,\Omega_t]$$

$$-E[\xi+\int_t^{t_2}\tilde f(v,X_v)dv+\int_t^{t_2}\tilde g(v,X_v)d\langle B\rangle_v\,|\,\Omega_t]$$

$$\le E[\int_t^{t_2}[\tilde f(v,L_v)-\tilde f(v,X_v)]dv+\int_t^{t_2}[\tilde g(v,L_v)-\tilde g(v,X_v)]d\langle B\rangle_v\,|\,\Omega_t]$$

$$=E[\int_t^{t_2}[r(v,\ L_v)-r(v,\ X_v)]dv+\int_t^{t_2}[g(v,p(v,L_v))-g(v,p(v,X_v))]d\langle B\rangle_v\,|\,\Omega_t]$$

But $p(t,\ L_t)=L_t$ gives $r(t,\ L_t)=0$. Also $p(t,\ X_t)=L_t$ and $X_t<L_t$ gives $r(t,\ X_t)=L_t-X_t>0$ in $(t_1,\ t_2)$. Thus

$$L_t-X_t\le E[-\int_t^{t_2}r(v,\ X_v)dv\,|\,\Omega_t]\le 0, \qquad (4.3)$$

which is a contradiction. Hence $L_t\le X_t$ for $t\in[0,\ T]$. Now assume that there exists an interval $(t_1,\ t_2)\subset[0,\ T]$ such that $X_{t_2}=U_{t_2}=\zeta,\ U_t<X_t$ for $t\in[0,\ T]$.

Then

$$X_t-U_t=E[\zeta+\int_t^{t_2}\tilde f(v,\ X_v)dv+\int_t^{t_2}\tilde g(v,\ X_v)d\langle B\rangle_v\,|\,\Omega_t]$$

$$-E[\zeta+\int_t^{t_2}\tilde f(v,\ U_v)dv+\int_t^{t_2}\tilde g(v,\ U_v)d\langle B\rangle_v\,|\,\Omega_t]$$

$$\le E[\int_t^{t_2}[\tilde f(v,\ X_v)-\tilde f(v,\ U_v)]dv+\int_t^{t_2}[\tilde g(v,\ X_v)-\tilde g(v,\ U_v)]d\langle B\rangle_v\,|\,\Omega_t]$$

$$=E[\int_t^{t_2}[r(v,\ X_v)-r(v,\ U_v)]dv+\int_t^{t_2}[g(v,\ p(v,X_v))-g(v,p(v,\ U_v))]d\langle B\rangle_v\,|\,\Omega_t].$$

Since $p(t,\ U_t)=U_t$ gives $r(t,\ U_t)=0$ but $p(t,\ X_t)=U_t$ yields $r(t,\ X_t)=U_t-X_t<0$ in $(t_1,\ t_2)$. Hence

$$X_t-U_t\le E[\int_t^{t_2}r(v,\ X_v)dv\,|\,\Omega_t]\le 0,$$

which is again a contradiction. Thus $X_t\le U_t$ for $t\in[0,\ T]$.

## EXISTENCE OF SOLUTION FOR G-BSDEs WITH DISCONTINUOUS DRIFT COEFFICIENT

We now consider the following backward stochastic differential equation under G-Brownian motion (G-BSDE)

$$X_t=E[\xi+\int_t^T f(v,X_v)dv+\int_t^T g(v,X_v)d\langle B\rangle_v\,|\,\Omega_t],t\in[0,T], \qquad (5.1)$$

such that the coefficient $f(t,\ x)$ is not continuous. But suppose that it is increasing, that is, if $x\ge y$ then $f(t,\ x)\ge f(t,\ y)$ and $h$ is Lipschitz continuous (where the inequalities are interpreted component wise).

**Theorem 5.2.** Suppose that

(1). The mapping $f(t,\ x)$ is increasing and $g(t,\ x)$ is Lipschitz continuous in $x$.

(2). $L_t$ and $U_t$ are the respective lower and upper solutions of the G-BSDE (5.1) with $\int_t^T E[|f(L_v)|]dv<\infty$, $\int_t^T E[|f(U_v)|]dv<\infty$ and $L_t\le U_t$ for $t\in[0,\ T]$.

Then there exists at least one solution $X_t\in M_G^1(0,\ T;\ R^n)$ of the G-BSDE (5.1) such that $L_t\le X_t\le U_t$ for $t\in[0,\ T]$ q.s.

**Proof.** We define the space of all d-dimensional stochastic processes by $H$, that is, $H=\{X=\{X_t, t\in[0,T]\}: E[|X_t|]<\infty\}$ with the norm $\|X_t\|=\int_t^T E[|X_v|]dv$ for all $t\in[0,\ T]$, which is a Banach space, see chapter III page 40 of [15] or see [5,13,14].

Now we represent the order interval $[L_t,\ U_t]$ in $H$ by $K$, that is, $K=\{X_t : X_t\in H \,\&\, L_t\le X_t\le U_t\}$ for $t\in[0,\ T]$, which is closed and bounded by the above norm. By using the monotone convergence theorem [3], one can prove the convergence of a monotone sequence that belongs to $K$ in $H$. Thus $K$ is a regularly ordered metric space with the above norm. It is clear that for any process $V_t\in K$, $L_t$ and $U_t$ are the respective lower and upper solutions for the G-BSDE

$$X_t = E[\xi+\int_t^T f(v,V_v)dv+\int_t^T g(v,V_v)d\langle B\rangle_v \mid \Omega_t], t\in[0,T], \tag{5.2}$$

Thus by theorem 4.3, for any $X_T\in L_G^1(\Omega_T,\ R^n)$ with $E[|X_T|]<\infty$ and $L_T\le X_T\le U_T$, the G-BSDE (5.2) has a unique solution $X_t\in M_G^1(0,\ T;\ R^n)$ such that $L_t\le X_t\le U_t$ for $t\in[0,\ T]$.

Define an operator $F: K\to K$ by $F(V_t)=X_t$, where $X_t$ is the unique solution of the G-BSDE (5.2). We will use theorem 6.2 to show that $F$ has a fixed point, which is then the required solution. If we show that $F$ is an increasing mapping then it has a fixed point.

We have to prove that if $V_t^{(1)}$ and $V_t^{(2)}$ are stochastic processes in $K$ such that $V_t^{(1)}\le V_t^{(2)}$ then $X_t^{(1)}\le X_t^{(2)}$ for all $t\in[0,\ T]$, where $X_t^{(1)}=F(V_t^{(1)})$ and $X_t^{(2)}=F(V_t^{(2)})$.

Let $V_t^{(1)}\le V_t^{(2)}$ for all $t\in[0,\ T]$ and define $X_t^{(1)}=F(V_t^{(1)})$, $X_t^{(2)}=F(V_t^{(2)})$ where $V_t^{(1)}, V_t^{(2)}\in K$. It is given that the coefficient $f$ is an increasing function therefore $X_t^{(1)}$ is a lower solution of the G-BSDE

$$X_t = E[\xi+\int_t^T f(v,V_v^{(2)})dv+\int_t^T g(v,X_v)d\langle B\rangle_v \mid \Omega_t], t\in[0,T]. \tag{5.3}$$

But this problem has an upper solution $U_t$. Hence by theorem 4.3, the G-BSDE (5.3) has a solution $X_t^{(2)}$ such that $X_t^{(1)}\le X_t^{(2)}\le U_t$. Thus $F$ is an increasing mapping and by theorem 6.2, it has a fixed point $X_t^{(*)}=F(X_t^{(*)})\in K$ such that $L_t\le X_t^{(*)}\le U_t$, where

$$X_t^{(*)}=E[\xi+\int_t^T f(v,\ X_v^{(*)})dv+\int_t^T g(v,\ X_v^{(*)})d\langle B\rangle_v \mid \Omega_t].$$

Now continuing example (3.3), the scalar G-BSDE (3.3) has at least one solution $X^{(*)}$ such that $L_t\le X_t^{(*)}\le U_t$, where $L_t$ and $U_t$ are the respective lower and upper solutions of the G-BSDE (3.3) given in the mentioned example.

## APPENDIX

For the following definition and theorem see [10].

**Definition 6.1.** An ordered metric space $M$ is called regularly (resp. fully regularly) ordered, if each monotone and order (resp. metrically) bounded ordinary sequence of $M$ converges.

**Theorem 6.2.** If $[a,\ b]$ is a nonempty order interval in a regularly ordered metric space, then each increasing mapping $F:[a,\ b]\to[a,\ b]$ has the least and the greatest fixed point.

Consider the G-BSDE (3.7) and define a mapping $Z_t: M_G^1(0,\ T;\ R^n)\to M_G^1(0,\ T;\ R^n)$ on a fixed interval [0, T] by

$$Z_t(X)=E[\xi+\int_t^T f(v,X_v)dv+\int_t^T g(v,X_v)d\langle B\rangle_v \mid \Omega_t], t\in[0,T].$$

For the proof of the following lemma see [13, 14].

**Lemma 6.4.** For each $\eta\in M_G^{1,0}(0,\ T)$,

$$E[|Q_{0,T}(\eta)|]=E[|\int_0^T \eta_v d\langle B\rangle_v|]\le\int_0^T E[|\eta_v|]dv. \tag{6.1}$$

Thus $Q_{0,T}: M_G^{1,0}(0,\ T)\to L^1(\Omega_T Z$ is a linear continuous mapping and consequently it can be uniquely extended to $L_G^1(0,\ T)$. For $\eta\in M_G^1(0,\ T)$, it is denoted as usual

$$Q_{0,T}(\eta)=\int_0^T \eta_v d\langle B\rangle_v$$

and still we have

$$E[|Q_{0,T}(\eta)|]=E[|\int_0^T \eta_v d\langle B\rangle_v|]\le \int_0^T E[|\eta_v|]dv.$$

First we prove an important lemma which will be used in the next existence and uniqueness theorem, see [18].

**Lemma 6.5.** For each $X,\ \hat{X}\in M_G^1(0,\ T;\ R^n)$ we have the following estimate;

$$E[|Z_t(X)-Z_t(\hat{X})|]\le C\int_0^t E[|X_v-\hat{X}_v|]dv, \qquad t\in[0,\ T], \qquad (6.2)$$

where $C$ is an arbitrary constant, depends only on the Lipschitz constant $K$.

**Proof.** Since $f$ and $g$ are Lipschitz continuous functions and using lemma 6.4 we have

$$E[|Z_t(X)-Z_t(\hat{X})|]\le E[\int_t^T |\tilde{f}(v,\ X_v)-\tilde{f}(v,\ \hat{X}_v)|dv$$

$$+\int_t^T |\tilde{g}(v,\ X_v)-\tilde{g}(v,\ \hat{X}_v)|d\langle B\rangle_v]$$

$$\le E[K\int_t^T |X_v-\hat{X}_v|dv+K\int_t^T |X_v-\hat{X}_v|d\langle B\rangle_v]$$

$$\le K\int_t^T E[|X_v-\hat{X}_v|]dv+K\int_t^T E[|X_v-\hat{X}_v|]dv$$

$$=2K\int_t^T E[|X_v-\hat{X}_v|]dv=C\int_t^T E[|X_v-\hat{X}_v|]dv,$$

where $C=2K$ is an arbitrary constant. Hence it is the required result.

**Theorem 6.6.** The backward stochastic differential equation (3.7) has a unique solution $X_t\in M_G^1(0,\ T;\ R^n).$

**Proof.** To show that the G-BSDE (3.7) has a unique solution we prove that $\Lambda(X)$ is a contraction mapping. Multiplying (6.2) by $e^{2Ct}$ and integrating on $[0,\ T]$ yields

$$\int_0^T E[|Z_t(X)-Z_t(\hat{X})|]e^{2Ct}dt\le C\int_0^T\int_t^T E[|X_v-\hat{X}_v|]e^{2Ct}dvdt$$

$$=C\int_0^T E[|X_v-\hat{X}_v|]\int_0^v e^{2Ct}dtdv$$

$$=\frac{1}{2}C\int_0^T (e^{2Cv}-1)E[|X_v-\hat{X}_v|]dv$$

$$\le\frac{1}{2}\int_0^T e^{2Cv}E[|X_v-\hat{X}_v|]dv.$$

Thus

$$\int_0^T E[|Z_t(X)-Z_t(\hat{X})|]e^{2Ct}dt\le\frac{1}{2}\int_0^T e^{2Ct}E[|X_t-\hat{X}_t|]dt. \qquad (6.3)$$

One can observe that the following two norms are equivalent in $M_G^1(0,\ T;\ R^n)$, i.e.,

$$\int_0^T E[|X_t|]dt\sim\int_0^T E[|X_t|]e^{2Ct}dt.$$

Hence from (6.3) we get that $Z_t(X)$ is a contraction mapping.

## CONCLUSION

Upper and lower solutions method is a very useful technique for the existence theory of boundary value problems (BVP). This method is widely used in ordinary and partial differential equations [3, 6, 12]. But a very limited literature is available on the method of upper and lower solutions for stochastic differential equations [8, 9]. The mentioned method for stochastic differential equations under G-Brownian motion (G-SDEs) was established by Faizullah and Piao in [5]. Furthermore, this is still an open problem to develop the method of upper and lower solutions for classical backward stochastic differential equations.

## ACKNOWLEDGEMENTS

The research of F. Faizullah is supported by the China Scholarships Council (CSC) and partially by National University of Sciences and Technology (NUST) Pakistan. We are very grateful to Prof. Daxiong Piao for his motivations and some useful suggestions for this work. We also thank to the anonymous reviewers for their valuable suggestions.

Authors acknowledge the immense help received from the scholars whose articles are cited and included in references of this manuscript. The authors are also grateful to authors / editors / publishers of all those articles, journals and books from where the literature for this article has been reviewed and discussed.

## REFERENCES

1. Assing S., Manthey R. The behaviour of solutions of stochastic differential

inequalities. Probability Theory and Related Fields, 1995; **103:** 493-514.
2. Bai X., Lin Y. On the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion with Integral-Lipschitz coefficients, 2010; arXiv:math.PR/1002.1046v2.
3. Cabada A., Habets P., Lois S. Monotone method of the neumann problem with lower and upper solutions in the reverse order, Appl. Math. Comput., 2001; **117:** 1-4.
4. Denis L., Hu M., Peng S. Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths, Potential Anal. 2010; **34:** 139-161.
5. Faizullah F., Piao D. Existence of solutions for G-SDEs with upper and lower solutions in the reverse order, 2012; 7**(3):** 432-439.
6. Graef J. R., Kong L., Minhos F. M., Fialho J. On the lower and upper solution method for higher order functional boundary value problems, Appl. Anal. Discrete Math., 2011; **5:** 133-146.
7. Gao F. Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion. Stochastic Processes and their Applications 2009; **10**: 3356-3382.
8. Halidias N., Kloeden P. E. A note on strong solutions of stochastic differential equations with a discontinuous drift coefficient, Journal of Applied Mathematics and Stochastic Analysis, 2006.
9. Halidias N., Michta M. The method of upper and lower solutions of stochastic differential equations and applications, Stochastic Analysis and Applications, 2008; **26**: 16-28.
10. Heikkila S., Hu S. On fixed points of multifunctions in ordered spaces, Applicable Analysis 1993; **51:** 115-127.
11. Heikkila S., Lakshmikantham V. Monotone iterative techniques for discontinuous nonlinear differential equations, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Newyork 1994; vol. 181.
12. Khan R.A., Faizullah F., Rafique M. Existence and approximations of solutions of boundary value problems on time scales, Adv. Dyn. Syst. Appl., 2009; **4:** 197-209.
13. Ladde G. S., Lakshmikantham V. Random differential inequalities, Mathematics in Science and Engineering, Acadmic Press, Newyork, 1980; vol. 150.
14. Li X., Peng S. Stopping times and related Ito's calculus with G-Browniain motion, Stochastic Processes and their Applications. (doi: 10.1016/j.spa. 2011.03.009) .
15. Nutz M., Soner H. M. Superhedging and dynamic risk measures under volatility uncertainty arXiv:math.PR/:1011.2958v1, 2010.
16. Peng S. G-expectation, G-Brownian motion and related stochastic calculus of Ito's type, The Abel Symposium 2005, Abel Symposia 2, Edit. Benth et. al. Springer-Vertag. 2006; 541-567.
17. Peng S. Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, Stochastic Processes and their Applications, 2008; **12**: 2223-2253.
18. Peng S. Nonlinear expectations and stochastic calculus under uncertainty. First Edition, arXiv:math.PR/:1002.4546v1, 2010.
19. Song Y. Properties of hitting times for G-martingale and their applications, Stochastic Processes and their Applications. (doi: 10.1016/j.spa. 2011.04.007).
20. Zhang D. Stability theorems for stochastic differential equations driven by G-Brownian motion. arXiv:math.PR/1105.4222v1, 2011.