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STABILITY OF A THERMO-CONVECTIVE PROBLEM UNDER THE INFLUENCES OF VARIABLE GRAVITY, INTERNAL HEAT SOURCE AND ROTATION

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ABSTRACT

Objective: The purpose of the present study is to discuss the stability of a thermo-convective problem where the gravity field varies across the layer under the influences of internal heat source and rotation when both the boundary surfaces are free. **Methods:** Linear stability analysis is performed to discuss the stability of the problem and the associated eigen-value problem is solved by using Newton-Raphson method. **Results:** The numerical approximations of the Rayleigh number are obtained for various values of the parameters and presented in tables 1 and 2. Numerical results show a destabilizing effect of the presence of heat source. Comparisons with the available published work are also presented. **Conclusion:** It is found that the critical Rayleigh number increases with an increase in gravity field and decrease in heat source i.e. the region of stability decreases with an increase in heat source.
Keywords: Thermal convection, Eigen-value problem, Variable gravity, Heat source

INTRODUCTION

Natural (Free) convection in a horizontal layer of fluid heated from below has been the subject of study for many decades owing to its importance in several engineering applications such as Chemical Engineering, Geophysics, Oceanography and Thermal Insulation Systems. The earliest experiments to demonstrate the onset of thermal instability in thin liquid layers are those of Bénard (1900). Rayleigh (1916) was the first to study the problem theoretically and apply the method of perturbations. A detailed amount of thermal convection in a Newtonian fluid layer in the presence of magnetic field and rotation has been given by Chandrasekhar (1981). Energy stability theory has been enlarged by the work of Serrin (1959), Joseph (1965, 1966), (Galdi & Straughan, 1985), (Galdi & Padula 1990), (Straughan 2004). Mulone and Rionero (1989) have studied nonlinear stability of a rotating Bénard problem via the Lyapounov direct method. Straughan (1989) has studied the convection in a variable gravity field in a viscous fluid layer. Further extensions of this problem by considering the effects of porous medium and internal heat source have been investigated by Rionero and Straughan (1990). Qin and Kaloni (1995) have studied nonlinear stability of a rotating Bénard problem in a porous medium by using energy theory. Straughan (2001) has studied the effect of a sharp nonlinear stability threshold in rotating porous convection. Nonlinear convection in a porous medium with inclined temperature gradient and variable gravity effects have been performed by Qiao and Kaloni (2001). Alex and Patil (2002a) have discussed the effect of a variable gravity field on convection in an anisotropic porous medium with internal heat source and inclined temperature gradient. Keeping in mind the importance of various parameters like variable gravity, thermal convection, internal heat source and rotation in Geophysics, Crystal Growth and Earth's Sciences, our interest in the present paper is to find the stability of a horizontal layer of fluid heated from below for a variable gravity field under the effects of non-uniform heat source and coriolis force. As is well known, variable gravity fields are of likely importance in convective flows and also in material processing technology.

RESEARCH METHODOLOGY

The following Research Methodology is adopted for the proposed Research paper:

- Identification of the problem
- Collection and study of related literature
- Mathematical formulation of the problem
- Stability analysis and numerical solution of the mathematical model
- Interpretation of results
- Conclusion

Mathematical formulation of the problem

The physical setting for our problem is as follows. We consider the motion of a heat conducting viscous fluid in a channel of infinite length and height d. The fluid layer is acted upon by a variable gravity field acting in the z-direction and is orthogonal to the fluid layer with a non-uniform heat source Q (z) and Coriolis force.

Under the Boussinesq approximation, the equations governing the convective motion are

$$\frac{\partial u_i}{\partial t} + (u_i.grad)u_i = -\frac{1}{\rho_0} grad p + \upsilon \Delta u_i - (1 + \frac{\delta\rho}{\rho_0}) g[1 + \varepsilon h(z)]k_i + 2\varepsilon_{ijk} u_j \Omega_k \qquad (2.1)$$

$$div\, u_i = 0 \qquad (2.2)$$

$$\frac{\partial T}{\partial t} + \left(u_j.grad\right)T = \kappa \Delta T + Q(z) \qquad (2.3)$$

Where υ is the coefficient of kinematic viscosity, ρ the density, k the thermal diffusivity, p the pressure, T the temperature, Q(z) the non-uniform internal heat source and $\varepsilon h(z)$ represents gravity variations.

The equation of state is $\rho = \rho_0[1 + \alpha(T_0 - T)]$ (2.4)

Linear stability analysis and solution for eigenvalue problem

To investigate the linear stability, the governing equations (2.1) - (2.3) in non-dimensional form (Omitting the star over each variable hereafter for the sake of convenience) can be written as

$$\frac{\partial u_i}{\partial t} = -grad\ p + H(z) R\theta k_i + \Delta u_i + \varepsilon_{ijk} T_A u_j \delta_{k3} \qquad (3.1\ a)$$

$$div\, u_i = 0 \qquad (3.1\ b)$$

$$P_r \frac{\partial \theta}{\partial t} + P_r u_j \frac{\partial \theta}{\partial x_j} = RN(z) w + \Delta\theta \qquad (3.1\ c)$$

Where $u = (u, v, w),\ N(z) = 1 + \delta q(z),\ H(z) = 1 + \varepsilon h(z),\ h(z) = -kz,\ \varepsilon, \delta \in \left(0, \frac{1}{k}\right),\ z \in (0,1),\ k = 1, 2, \ldots$

R denotes the Rayleigh number and θ is the perturbed temperature.

The boundary conditions are $u_i = 0, \quad \theta = 0 \quad at \quad z = 0\ and\ \ d.$ (3.2)

We introduce the following non-dimensional quantities in equations (3.1a) – (3.1c)

$$x = x^* d, \quad t = t^* \frac{d^2}{\upsilon}, \quad u = u^* \frac{\upsilon}{d}, \quad p = p^* \rho_0 \frac{\upsilon^2}{d^2}, \quad T_A = T_A^* \left(\frac{2 d^2 \Omega}{\upsilon} \right)$$

$$T = T^* \left(\frac{\beta \upsilon^3}{g \alpha \kappa d^2} \right)^{\frac{1}{2}}, \quad \theta = \theta^* \left(\frac{\beta \upsilon^3}{g \alpha \kappa d^2} \right)^{\frac{1}{2}}, \quad \delta q(z) = \frac{F(z)}{c}$$

And the following dimensionless numbers are defined by

$$R^2 = \frac{g \alpha \beta d^4}{\kappa \upsilon} \qquad (Rayleigh\ number)$$

$$P_r = \frac{\upsilon}{\kappa} \qquad (\mathrm{Pr}andtl\ number)$$

$$T_A = \left(\frac{4 \Omega^2 d^4}{\kappa \upsilon} \right) \qquad (Taylor\ number)$$

$$F(z) = \frac{1}{k} \int_0^z Q(\xi) d\xi$$

Eliminating δp between equations (3.1 a) and (3.1 c), we find (after using the boundary conditions and divergence theorem)

$$\frac{d}{dt}\left(\frac{1}{2} \|u\|^2 \right) = R \left\langle H(z)\theta, w \right\rangle - \|\nabla u\|^2 + T^2 \|w_{,z}\|^2 \tag{3.3}$$

$$\frac{d}{dt}\left(\frac{1}{2} P_r \|\theta\|^2 \right) = R \left\langle N(z)\theta, w \right\rangle - \|\nabla \theta\|^2 \tag{3.4}$$

Here, V denotes a periodicity cell, $\langle . \rangle$ denotes the integration over V, and $\|.\|$ denotes the $L^2(V)$ norm.

The system of equations (3.3) and (3.4) can be put in the form $\frac{dE}{dt} = RI - D$ (3.5)

Where the Energy functional E, the production term I, and the dissipation term D are defined as: $E = \left(\frac{1}{2}\|u\|^2 + \frac{1}{2} P_r \|\theta\|^2 \right)$ $I = \left(\left[H(z) + N(z) \right] \theta, w \right)$ and $D = \left(\|\nabla u\|^2 + \|\nabla \theta\|^2 + T^2 \|w_{,z}\|^2 \right)$ (3.6)

We now defined $\frac{1}{\mathrm{R_E}} = \max \frac{\mathrm{I}}{\mathrm{D}}$ (3.7)

On combining (3.5) with (3.6) and (3.7), and by using Poincare' inequality $\|\nabla \theta\|^2 \geq \mu \|\theta\|^2$, we can infer that $\frac{dE(t)}{dt} \leq -2m\mu E(t)$ (3.8)

And $m = \frac{R_E - R}{R_E} > 0$ when $\mathrm{R} < \mathrm{R_E}$ (3.9)

Inequality (3.8) clearly indicates that $E(t) \to 0$ at least exponentially as $t \to \infty$.

We now return to (3.7) and consider the Variational problem for the determination of $\mathrm{R_E}$. The associated Euler-Lagrange's equations become

$$2\nabla^2 u + R_E \left[M(z) \right] \theta k_i + 2T^2 w_{,zz} = p_{,i} \tag{3.10}$$

$$\nabla . u = 0 \tag{3.11}$$

$$2\nabla^2\theta + R_E\left[M(z)\right]w = 0 \tag{3.12}$$

Where $w = u_i k_i$ is the z-component of the velocity, p is a Lagrange's multiplier introduced, since u_i is solenoidal.

On taking curlcurl of equation (3.10), we find the equations in w and θ of the form,

$$2\Delta^2 w - R_E\left[M(z)\right]\Delta^*\theta + 2T^2\Delta w_{,zz} = 0 \tag{3.13}$$

$$2\Delta\theta + R_E\left[M(z)\right]w = 0 \tag{3.14}$$

Where $\Delta^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and M(z)=H(z)+N(z)

Eliminating θ between equations (3.13) and (3.14), we find

$$\left(D^2 - k^2\right)^3 w = \frac{R_E{}^2 k^2\left[M(z)\right]^2 w + 2T^2\Delta w_{,zz}}{4} \tag{3.15}$$

Where $D = \frac{d}{dz}$, $R_E{}^2$ the Rayleigh number for energy stability and k is the wave number.

In the absence of rotation, variable gravity and internal heat source [i.e. T=0, H(z)=1, N(z)=1], the problem becomes the classical problem of convection [3] and the critical Rayleigh number is $R_{EC}{}^2 = \frac{27\pi^4}{4} \approx 657.513$. Hence for a Rayleigh number less than 657.513, we have stability i.e. all perturbations decay to zero. On the other hand, if the Rayleigh number is greater than 657.513, we obtain instability.

Thus the exact solution w of equation (3.15) subject to the boundary conditions $w = 0\ ,\ \theta = 0\ ,\ D^{2n}(w) = 0\ \ at\ \ z = 0\ \ and\ \ d$ (free boundary case) can be written in the form $w = \sin n\pi z$ (for the lowest mode). (3.16)

Substituting solution (3.16) into equation (3.15), we obtain

$$R_E{}^2 = \frac{4\left(n^2\pi^2 + k^2\right)^3 + 4n^2\pi^2T^2\left(n^2\pi^2 + k^2\right)}{\left[H(z) + N(z)\right]^2 k^2} \tag{3.17}$$

Considering $k^2 = n^2\pi^2 x$, we get $R_E{}^2 = \frac{4n^4\pi^4\left[(1+x)^3 + \frac{T^2(1+x)}{n^2\pi^2}\right]}{x.\left[H(z) + N(z)\right]^2}$ (3.18)

As a function of x, R_E given by equation (3.18) attains its minimum when

$$2x^3 + 3x^2 = 1 + \frac{T^2}{n^2\pi^2} \tag{3.19}$$

With x determined as a solution of cubic equation (3.19), equation (3.18) will give the required critical Rayleigh number $R_{EC}{}^2$.

RESULTS AND DISCUSSION

The numerical results are presented for critical Rayleigh numbers $R_{EC}{}^2$ and wave numbers for different values of parameters when both bounding surfaces are free.

Values of R_{EC}^{2} determined for various values of T^2, H(z) and N(z):

T^2	k^2	ε	h(z)	δ	q(z)	R_{EC}^{2}
0	4.943	0	0	0	0	657.513
10	6.707	0.2	-0.1	0.2	1	776.862
50	10.699	0.4	-0.2	0.4	5	458.689
100	13.702	0.6	-0.3	0.6	10	173.580
200	17.727	0.8	-0.4	0.8	15	91.022
500	25.029	1.0	-0.5	1.0	20	74.236

Table 1 : Rayleigh number for different values of the parameters ε, k, T, δ, $h(z)$, $q(z)$, When n = 1

T^2	k^2	ε	h(z)	δ	q(z)	R_{EC}^{2}
0	4.943	0	0	0	0	657.513
10	21.831	0.2	-0.1	0.2	1	9806.738
50	28.356	0.4	-0.2	0.4	5	4094.849
100	34.131	0.6	-0.3	0.6	10	1321.316
200	42.775	0.8	-0.4	0.8	15	602.600
500	59.516	1.0	-0.5	1.0	20	425.164

Table 2 : Rayleigh number for different values of the parameters ε, k, T, δ, $h(z)$, $q(z)$, When n = 2

The approximate values of the Rayleigh number for various values of $\varepsilon, k, \delta, T, h(z), q(z)$ for the case, when both the boundaries are free, are presented in Table 1 (n=1), Table 2 (n=2).

Tables 1 and 2 represent that the values of critical Rayleigh number are decreasing rapidly in both cases with an increasing heat source. The table also shows that the increase in ε (i.e. the decrease in $h(z)$) and in $q(z)$ reduces the domain of stability.

For $\varepsilon = 0$, $\delta = 0$, $T = 0$ **,** we regain the known [3] critical values of Rayleigh number, $R_{EC}^{2} = 657.513$ **,** $R_{EC}^{2} = 657.513$ for n=1, n=2 respectively. (for free boundaries)

CONCLUSION

In this paper, for a horizontal layer of fluid heated from below in the case of variable gravity in z-direction with a non-uniform heat source and rotation, we obtained analytical expression of the Euler-Lagrange equation and perform numerical computations for the case of free boundaries. The Newton-Raphson method is used to compute the values of critical Rayleigh number for various modes. The table shows the influences of each parameter on critical Rayleigh number i.e. the region of stability decreases with the increase in heat source and decrease in gravity field. In the absence of heat source and rotation, the numerical results obtained are in good agreement with the previous published work [15], [16].

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