IJCRR
Vol 05 issue 15
Section: Technology
Category: Research
Received on: 04/07/13
Revised on: 22/07/13
Accepted on: 09/08/13

THERMOSOLUTAL INSTABILITY OF AN INCOMPRESSIBLE, VISCOUS FLUID CONFINED IN A POROUS MEDIUM IN THE PRESENCE OF MAGNETIC FIELD

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ABSTRACT

The paper critically examines, within the framework of linear analysis, thermosolutal instability of an incompressible, viscous fluid confined in a porous medium in the presence of magnetic field, analytically and numerically both. The analytical discussion provides the sufficient conditions of stability and instability and the characterization of modes. By actually calculating the root of eigenvalue equation (of degree 4) neutral stability curves are drawn. The numerical results show the effect of various physical parameters on the critical wave number a_c. It is concluded that R_D^{-1} , S and R_4 have stabilizing character and Richardson number J has destabilizing character. The non-dimensional parameter R_2 shows a dual character, which depends upon thermal diffusivity κ.

Keywords: Thermosolutal instability, porous medium, magnetic field

INTRODUCTION

The problems on thermal instability (Bénard convection) in a fluid layer under varying assumptions of hydrodynamics and hydromagnetic have been discussed in detail by Chandrasekhar[1] in his celebrated monograph. The problem of thermosolutal instability in fluids through a porous medium is of importance in geophysics, soil sciences, ground-water hydrology and astrophysics. The development of geothermal power resources holds increased general interest in the study of the properties of convection in a porous medium.The instability of fluid flows in a porous medium under varying assumptions has been well summarized by Scheidegger [2] and Yih [3]. While investigating the flows or flow instabilities through porous media, the liquid flow has been assumed to be governed by Darcy's Law [4] by most of the research workers, which neglects the inertial forces on the flow. Beavers *et al.* [5] demonstrated experimentally the existence of shear within the porous medium near surface, where the porous medium is exposed to a freely flowing fluid, thus forming a zone of shear-induced flow field. Darcy's equation however, cannot predict the existence of such a boundary zone, since no macroscopic shear term is included in this equation (Joseph and Tao [6]). To be mathematically compatible with the Navier-Stokes equations and physically consistent with the experimentally observed boundary shear zone mentioned above, Brinkman [7] proposed the introduction of the term $-\left(\frac{\mu}{k_1}\right)\boldsymbol{V}$ in addition to $\frac{\mu}{\varphi}\nabla^2\boldsymbol{V}$ in the equations of fluid motion. The thermosolutal convection in a porous medium was studied by Nield and Bejan [8].

Instability of compressible or incompressible flow has been studied extensively by a number of research workers in past few decades. In almost all such investigations, the Boussines'q approximation is used to simplify the equations of motion. Goel *et.al*. [9] examined the shear flow instability of an incompressible visco-elastic second order fluid in a porous medium in which the modified Darcy's law is replaced by the celebrated Brinkman model so that both the inertia and viscous terms are included in their usual forms.

The behaviour of conducting fluid is very much different in the absence and in the presence of a magnetic field. The interesting properties associated with a magnetic field, have attracted a number of research workers to work in this direction. Bansal and Agrawal [10] have studied the thermal instability of a compressible shear flow in the presence of a weak applied magnetic field. The problem for compressible shear layer in the presence of a weak applied magnetic field through porous medium has been studied by Bansal *et.al.* [11].

In the present paper, an attempt has been made to examine the thermosolutal instability of an incompressible, viscous fluid in the presence of magnetic field and confined in a porous medium following Brinkman model. The Boussinesq approximation is used throughout the paper. It states that variations of density in the equations of motion can safely be ignored everywhere except its association with the external force. The approximation is well justified in the case of incompressible fluids.

FORMULATION OF THE PROBLEM

Here we consider an infinite, horizontal, incompressible, viscous fluid saturating an isotropic porous medium and which is subjected to uniform magnetic field in the horizontal direction. Uniform temperature and concentration gradients are maintained along z-axis. Equations expressing the conservation of momentum, mass, magnetic field, temperature, solute mass concentration and equation of state in Brinkman model are: $\frac{\rho}{\phi}\left[\frac{\partial \boldsymbol{V}}{\partial t}+\frac{1}{\phi}(\boldsymbol{V}.\nabla)\boldsymbol{V}\right]=-\nabla p+\boldsymbol{g}\rho+\mu\left(\frac{1}{\phi}\nabla^2-\frac{1}{k_1}\right)\boldsymbol{V}+\frac{1}{4\pi}(\nabla\times\boldsymbol{H})\times\boldsymbol{H}$

(1) $\nabla.\boldsymbol{V}=0$ (2)

$\nabla.\boldsymbol{H}=0$ (3)

$$\frac{\partial \boldsymbol{H}}{\partial t}=\frac{1}{\phi}\nabla\times(\boldsymbol{V}\times\boldsymbol{H}), \tag{4}$$

$$\frac{\partial T}{\partial t}+\frac{1}{\phi}(\boldsymbol{V}.\nabla)T=\kappa\nabla^2 T, \tag{5}$$

$$\frac{\partial C}{\partial t}+\frac{1}{\phi}(\boldsymbol{V}.\nabla)C=\kappa'\nabla^2 C \tag{6}$$

$$\rho=\rho_0\left[1-\alpha(T-T_0)-\alpha'(C-C_0)\right], \tag{7}$$

where $\boldsymbol{V}$, $\boldsymbol{H}$, ρ, μ, κ, κ', α, α', ϕ and k_1 are respectively fluid velocity, magnetic field intensity, density, viscosity coefficient, thermal diffusivity, solute diffusivity, thermal expansion coefficient, solute expansion coefficient, medium porosity and medium permeability. $\boldsymbol{g}=(0,0,-g)$ is the gravitational acceleration. Subscript zero in eq.(7) refers to the value at the reference level $z=0$.

Basic State

In the undisturbed state, the fluid is at rest. Constant temperature $\left(\beta=\left|\frac{dT}{dz}\right|\right)$ and concentration gradients $\left(\beta'=\left|\frac{dC}{dz}\right|\right)$ are maintained in the fluid and the uniform magnetic field acts in the horizontal direction (say, in the x-direction). Therefore, the basic state is described as $\boldsymbol{V}=(0,0,0), \boldsymbol{H}=(H_0,0,0), T=T_0-\beta z, C=C_0-\beta' z,\ \rho=\rho_0\left[1+(\alpha\beta+\alpha'\beta')z\right]$

and $$p=p_0-\rho_0 g\left[z+\frac{1}{2}(\alpha\beta+\alpha'\beta')z^2\right] \tag{8}$$

Perturbations and Normal Mode Analysis

The basic state characterized by eq.(8) is slightly perturbed, equations in perturbations are linearized within the framework of classical linear theory of stability, arbitrary perturbation *f'(x,y,z,t)* is analysed into normal modes as

$$f'(x,y,z,t)=f\ \exp\left[\sigma t+i(ax+by+cz)\right],$$

and the perturbation quantities are eliminated to yield the following eigen-value equation of degree four:

$$l^2\left[\frac{\sigma\rho_0}{\phi}+\mu\left(\frac{l^2}{\phi}+\frac{1}{k_1}\right)+\frac{H_0^2a^2}{4\pi\sigma\phi}\right](\sigma+\kappa l^2)(\sigma+\kappa' l^2)=\frac{m^2g\rho_0}{\phi}\left[\alpha\beta(\sigma+\kappa' l^2)+\alpha'\beta'(\sigma+\kappa l^2)\right] \tag{9}$$

where *a*, *b* and *c* are real and σ, a time constant, is complex, in general and $l^2 = a^2 + b^2 + c^2$ and $m^2 = a^2 + b^2$.

Restricting our discussion to two-dimensional disturbances in the *xy*-plane(so that *c = 0*) and introducing the transformations

$$\sigma = \frac{U_0 \sigma^*}{d} \quad \text{and} \quad (a,l) = \frac{1}{d}\left(a^*, l^*\right),$$

The final eigen-value equation(after dropping the asterisks) is obtained as

$$\sigma^4 + A\sigma^3 + B\sigma^2 + C\sigma + D = 0, \tag{10}$$

where $A = l^2\left(R_2 + R_3 + R_4\right) + R_D^{-1}$,

$$B = l^4\left(R_2R_3 + R_2R_4 + R_3R_4\right) + l^2\left(R_2 + R_3\right)R_D^{-1} + a^2S - \left(J + J'\right),$$

$$C = l^4 R_2 R_3\left(R_4 l^2 + R_D^{-1}\right) + \left(R_2 + R_3\right)a^2 l^2 S - l^2\left(JR_3 + J'R_2\right),$$

$$D = a^2 l^4 R_2 R_3 S,$$

$\upsilon = \dfrac{\mu}{\rho_0}$, $R_2 = \dfrac{\kappa}{dU_0}$, $R_3 = \dfrac{\kappa'}{dU_0}$, $R_4 = \dfrac{\upsilon}{dU_0}$, $R_D^{-1} = \dfrac{\upsilon d\phi}{k_1 U_0}$, $S = \dfrac{H_0^2}{4\pi\rho_0 U_0^2}$ is magnetic force number

and $J = \dfrac{g\alpha\beta d^2}{U_0^2}$ & $J' = \dfrac{g\alpha'\beta' d^2}{U_0^2}$ are Richardson numbers.

Analytical Discussion

Theorem1:- System is unstable under the condition $J + J' \geq Q$,

where $$Q = l^4\left(R_2R_3 + R_2R_4 + R_3R_4\right) + l^2\left(R_2 + R_3\right)R_D^{-1} + a^2S \tag{11}$$

Proof:- Under the condition of the theorem, the coefficient *B* of σ^2 becomes negative. If σ_k, *k = 1,2,3* and *4* are the roots of the eq.(10), then

$$\sum \sigma_1\sigma_2 = B < 0$$

It is clear from here, that either at least one root is positive or if all roots are complex, then exactly one pair has positive real parts. This ensures the instability of the system. The region of instability in (*J,J'*) plane is shown in *Fig.1*. This result is similar to the one obtained by Goel *et.al*. [12], however, the value of Q is different in this case from the value of Q in Goel *et.al.*

***Discussion for non-oscillatory modes*:**

Theorem2:- System is stable under the condition

$$\beta < 0 \text{ and } \beta' < 0. \tag{12a,b}$$

Proof:- Eq.(3.2) for non-oscillatory modes ($\sigma_i = 0$) becomes

$$\sigma_r^4 + A\sigma_r^3 + B\sigma_r^2 + C\sigma_r + D = 0 \tag{13}$$

If $J < 0$ and $J' < 0$, then the roots of eq.(13) are all negative, implying, thereby the stability of non-oscillatory modes.

The situation when both *J* and *J'* are negative will be referred to as potentially stable arrangement.

Theorem3:- If the condition (12a) is violated, then the system is stable under the condition

$$J \le \frac{\kappa}{\kappa'}|J'| \qquad \text{if} \quad \kappa < \kappa'$$

and $$J \le |J'| \qquad \text{if} \quad \kappa \ge \kappa'$$

Proof:- Proof is obvious.

*Remark***:-** A similar result can be proved when condition(12b) is violated. In this case, system is stable under the condition

$$J' \le \frac{\kappa'}{\kappa}|J| \qquad \text{if} \quad \kappa' < \kappa$$

and $$J' \le |J| \qquad \text{if} \quad \kappa' \ge \kappa$$

Theorem4:- If both the conditions (12a,b) are violated, then sufficient condition of stability is given by

$$S > \frac{J + J'}{a^2}$$

Proof:- Eq.(13) does not allow any positive root under the condition

$$S > max\left\{\frac{J + J'}{a^2}, \frac{JR_3 + J'R_2}{a^2(R_2+R_3)}\right\}$$

Since $J + J' > \frac{JR_3 + J'R_2}{(R_2+R_3)}$, therefore sufficient condition of stability is given by

$$S > \frac{J + J'}{a^2}.$$

This establishes a stabilizing role of magnetic field.

Theorem 5:- If $S < \frac{J + J'}{a^2}$, then system is stable under the condition

$$R_D^{-1} > \left[\frac{J + J'}{a^2} - S\right]\frac{a^2(R_2+R_3)}{l^2 R_2 R_3}$$

Proof:- For $S < \frac{J + J'}{a^2}$, from eq.(13), stability of the system follows under the conditions

$$R_D^{-1} > \left[\frac{J + J'}{a^2} - S\right]\frac{a^2}{l^2(R_2+R_3)}$$

and $$R_D^{-1} > \left[\frac{JR_3 + J'R_2}{a^2(R_2+R_3)} - S\right]\frac{a^2(R_2+R_3)}{l^2 R_2 R_3}$$

The above two conditions are replaced by the single condition

$$R_D^{-1} > \left[\frac{J + J'}{a^2} - S\right]\frac{a^2(R_2+R_3)}{l^2 R_2 R_3}$$

in view of the fact that

$$\frac{J + J'}{a^2} > \frac{JR_3 + J'R_2}{a^2(R_2+R_3)} \quad \text{and} \quad \frac{(R_2+R_3)}{R_2 R_3} > \frac{1}{(R_2+R_3)}$$

NUMERICAL RESULTS AND DISCUSSION

Eq.(10) is a fourth degree equation in σ with real coefficients, which depends upon R_D^{-1}, S, J, J', R_2, R_3, R_4, l and a. Our aim has been to examine the effect of various parameters such as R_D^{-1}, R_2, R_4, magnetic force number S and the Richardson number J on the stability of the system. This has been achieved by actually calculating the roots of the equation (3.2) correct upto three decimal places. The calculation of critical wave numbers has lead to the neutral stability curves.

Fig.2. shows the critical wave number a_c for different value of R_D^{-1} (curve-I). When R_D^{-1} =0, for a < 1.321 unstable modes are non-oscillatory i.e., the eigenvalue equation has all real roots with at least one positive root. For 1.321 < a < 2.153 unstable modes are oscillatory and system becomes stable for $a \geq 2.153$. As R_D^{-1} increases, a_c decreases so that the range of stable wave numbers increases. It is concluded that R_D^{-1} has a stabilizing character and a large value of R_D^{-1} is required to stabilize the system for all wave numbers. Curve-II separates the unstable modes into unstable oscillatory and unstable non-oscillatory modes. As R_D^{-1} increases, the range of both the unstable oscillatory and the unstable non-oscillatory modes decreases.

Fig.3. shows the stabilizing character of magnetic field (curve-I). Short wave length perturbations are more stable. The unstable modes are divided into oscillatory and non-oscillatory modes. The large wave length perturbations are unstable and non-oscillatory and the modes which are intermediatory between stable and unstable non-oscillatory modes are unstable and oscillatory, i.e., $a > a_c$ are stable, $a_c^* < a < a_c$ are unstable and oscillatory and a < a_c^* are unstable and non-oscillatory. It is important to observe that whereas the unstable modes are non-oscillatory in the absence of magnetic field, some oscillatory unstable modes are introduced in its presence .

Fig.4. shows the destabilizing character of J. As J increases, a_c increases which decreases the range of stable wave numbers. As is clear from this figure, the range of both the non-oscillatory unstable and the oscillatory unstable modes increases with J.

Fig.5. shows the dual character of the non-dimensional parameter R_2, which in fact, depends upon the thermal diffusivity κ. As R_2 increases from *0* to *0.3*, the range of stable wave increases. As R_2 increases from *0.3* to *3.5* approximately, the range of unstable wave numbers decreases rapidly, however, this decrease in the range of unstable wave numbers is slow as R_2 further increases beyond *3.5*. The neutral stability curve also shows that the critical wave number for maximum instability is given by $a_c = 2.081$ and it occurs for $R_2 = 0.3$. Similar character is exhibited for R_3.

Stabilizing character of fluid viscosity is exhibited in *Fig.6*. As R_4 increases, the range of stable wave numbers increases sharply upto $R_4 = 4$ (approx.) and then increases slowly as R_4 further increases.

CONCLUSION

The analytical discussion provides the sufficient conditions of stability and instability and the characterization of modes. Theorem 2 states that non-oscillatory modes are stable if $\beta < 0$ and $\beta' < 0$. This situation is known as potentially stable arrangement.

The numerical results show the effect of various physical parameters on the critical wave number a_c. On the basis of numerical discussion & neutral stability curves obtained in paper, it is concluded that medium porosity parameter R_D^{-1} , magnetic force number S and fluid viscosity have stabilizing character and Richardson number J has destabilizing character. The non-dimensional parameter R_2, which depends upon thermal diffusivity κ, shows a dual character,

ACKNOWLEDGEMENT

The author is grateful to Professor S.C. Agrawal, Retd. H.O.D., Department of Mathematics, C.C.S. University, Meerut, for providing valuable support.

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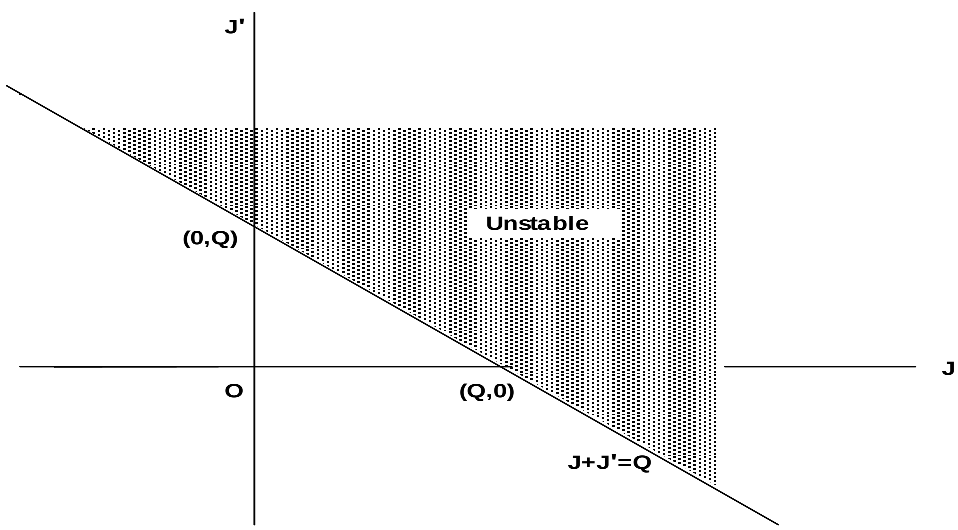

Fig.1. Unstable region (shaded) in (J,J') plane

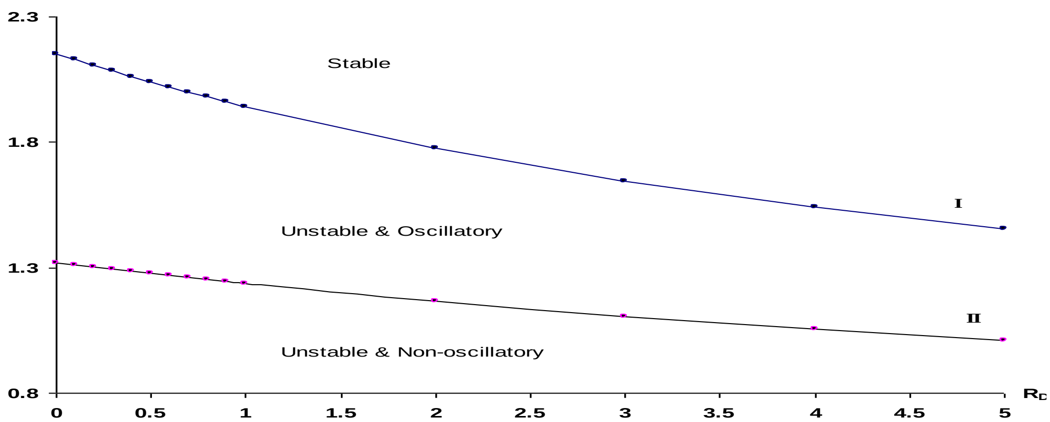

Fig.2. Critical wave number a_c Vs R_D^{-1} ($R_2 = R_3 = R_4 = 0.5$, J = J' = 5, S = 2)

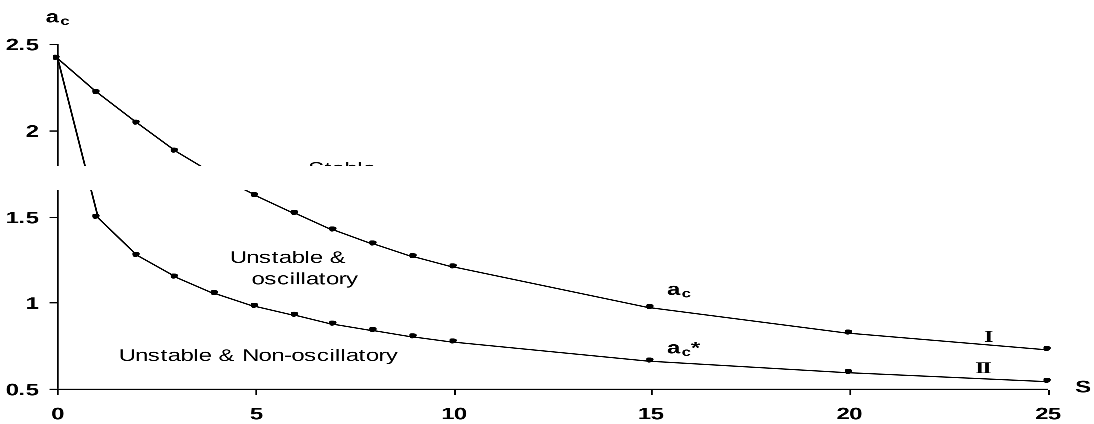

Fig.3. Critical wave number a_c Vs S ($R_2 = R_3 = R_4 = 0.5$, J = J' = 5, $R_D^{-1} = 0.5$)

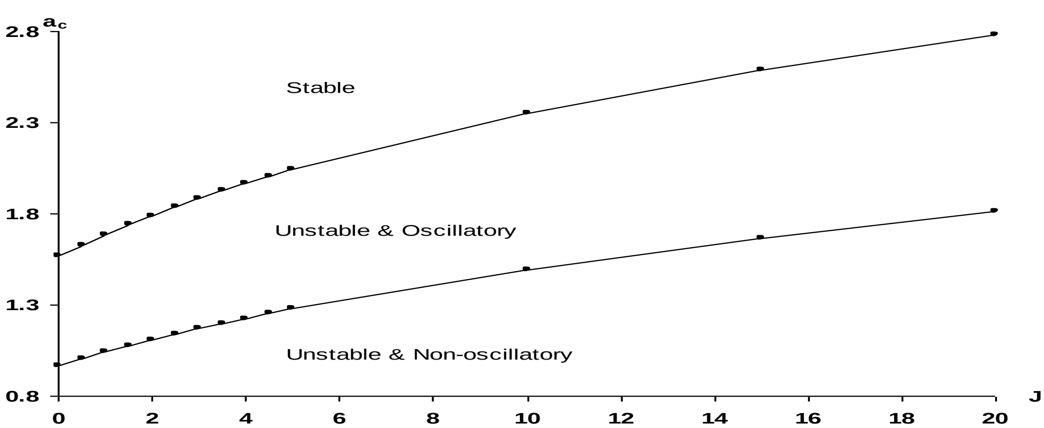

Fig.4. Critical wave number a_c Vs J ($R_2 = R_3 = R_4 = 0.5$, J' = 5, $R_D^{-1} = 0.5$,S = 2)

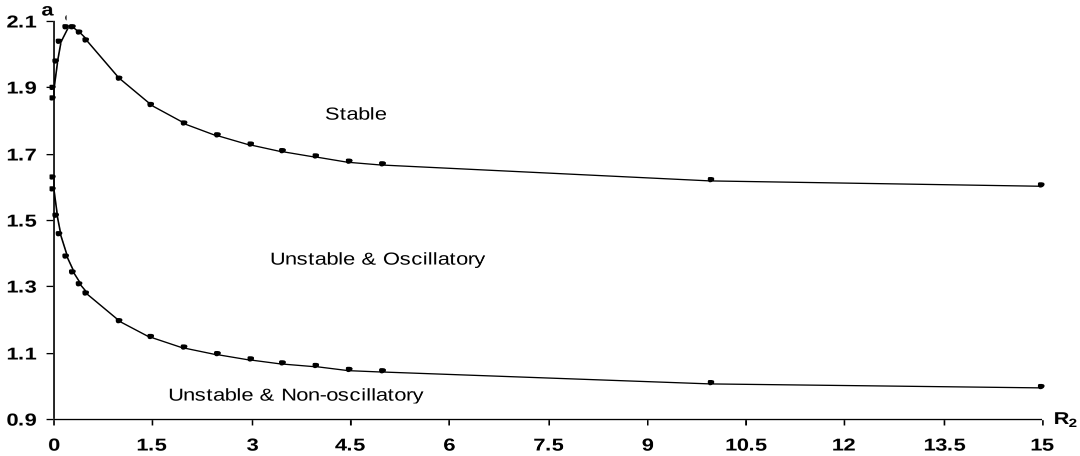

Fig.5. Critical wave number a_c Vs R_2 ($R_3 = R_4 = 0.5$, J = J' = 5, $R_D^{-1} = 0.5$,S = 2)

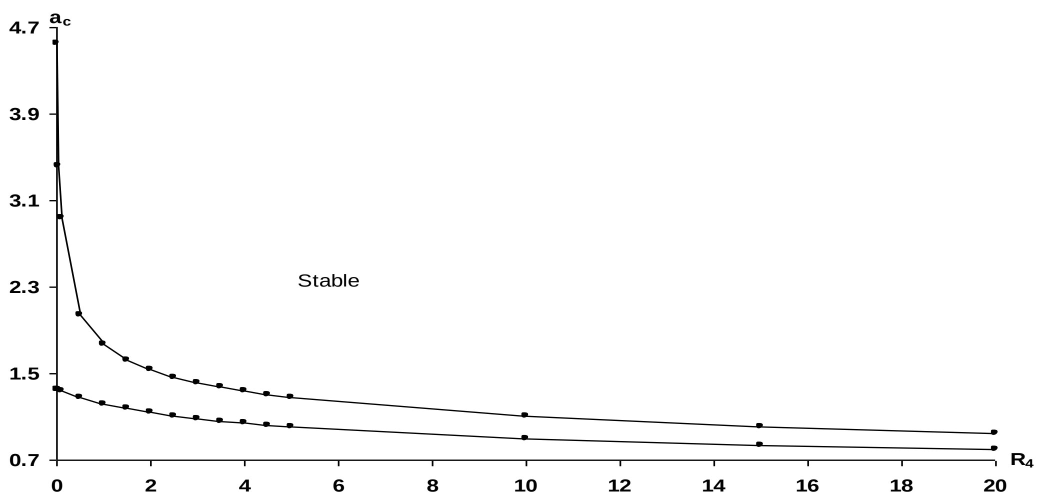

Fig.6. Critical wave number a_c Vs R_4 ($R_2 = R_3 = 0.5$, J = J' =5, $R_D^{-1} = 0.5$, S = 2)